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of their progress. Concise summaries in mimeographed form help greatly in fixing the grasp of the topics, especially as no text is available which covers any large part of the course from the same standpoint.

The treatment of topics is unconventional in that it keeps their practical uses steadily to the fore, and admits only topics, some of whose uses can be exhibited when the topic is studied. This gives students constant practice in applying their mathematics and in analyzing problems. It also creates considerable enthusiasm for the subject. Students see something of the meaning of mathematical work, and they do a very large amount of work very cheerfully—one may almost say eagerly. (Approximately one thousand exercises are worked during the year, many of which are fairly substantial problems.)

A word concerning the courses which follow. Naturally they must be modified somewhat. The second year is devoted to a systematic course in calculus, but several of the usual preliminaries, which were omitted from the first course, are treated incidentally as needed,—*e. g.*, elements of solid analytics before taking up multiple integration. Also, trigonometric analysis is reviewed in detail before differentiating trigonometric functions. Since the students are already familiar with the elements and general principles, they can be given an excellent grasp of the subject in this year's work. Besides covering a very full text-book treatment, they have time for some use of imaginaries in trigonometric reductions, for elementary problems on Fourier's series and calculus of variations, and for some practice in formulating as well as solving practical differential equations.

The more theoretical parts of college algebra and analytic geometry are postponed to the junior year when they can be dealt with adequately in connection with modern developments in algebra and geometry. Thus it is possible to devote the junior and senior years to mathematics of a fairly advanced type.

It seems to the writer that it would be perfectly feasible to give the freshman course outlined above in the fourth year of high schools and academies wherever there are strong teachers who know the subject well. Even then students would not be starting the study of these topics as young here as abroad. It is not hard to think of advantages which would result from their earlier introduction here.

LINKAGES.

By DICKSON H. LEAVENS, College of Yale in China, Changsha.

The subject of linkages is somewhat removed from the main lines of mathematics, but although it may not be of much importance in their development, it has considerable intrinsic interest. The problem that gave rise to the study of link-motions, that of making a point move in a straight line, is of significance theoretically in furnishing a method of drawing a straight line without begging the question as we do when we copy by means of a ruler a line already made.[1]

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[1] See A. B. Kempe, *How to Draw a Straight Line*, London, Macmillan, 1877, which also gives a very clear and simple account of linkages in general. It is perhaps the best introduction to the subject, and its footnotes give references to the original papers up to its date.

This problem is also of practical importance in machinery, where it is necessary to produce rectilinear motion of certain parts. The history of the subject, too, is interesting, three or four great minds have contributed to it, and some remarkable results have been obtained. Finally, it sometimes furnishes a striking way of illustrating graphically results obtained analytically.

A plane linkage may be defined as a system of bars (or more generally, plane figures) pivoted together so that they move in the same plane (*i. e.*, in the case of material links, in close parallel planes). A *complete* linkage will have an even number of bars, so connected that there is only one degree of freedom of motion. A *linkwork* consists of an odd number of bars, pivoted at two points to a fixed plane.[1] If we consider the fixed plane as a link, we have a linkage, so that word is often used loosely instead of linkwork.

Leaving out of consideration the simple and familiar pantograph, invented early in the seventeenth century by C. Scheiner, we may say that the subject had its origin with James Watt. In his improvements on the steam-engine he found the need of guiding the piston rod in a straight line. This rectilinear motion is now effected by having the cross-head slide on smooth-planed guides, but this could not be done with sufficient mechanical perfection in his day, and some method with less friction was required.

In 1784 Watt patented his so-called "parallel motion." In a letter[2] to his son, written twenty-four years later, he says: "The idea originated in this manner. On finding double chains, or racks and sectors, very inconvenient for communicating the motion of the piston-rod to the angular motion of the working-beam, I set to work to see whether I could contrive some means of performing the same from motions turning upon centers, and after some time it occurred to me that AB and CD (Fig. 1),[3] being two equal radii revolving on the centers B and C, and connected together by a rod AD, and moving together through arches of certain lengths, the variations from the straight line would be nearly equal and opposite, and that the point E would describe a curve nearly straight, . . .

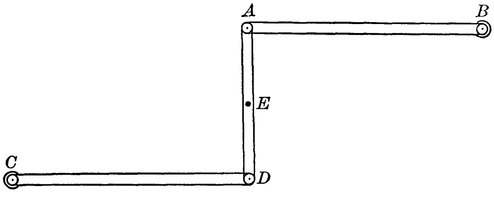

FIG. 1.

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[1] For these definitions and for future references to Sylvester, see *On Recent Discoveries in the Mechanical Conversion of Motion*, in Sylvester, *Collected Works*, Cambridge Univ. Press, 1909, III, pp. 7–25. This paper is one of the most interesting on the subject.

[2] J. P. Muirhead, *The Life of James Watt*, New York, Appleton, 1859, p. 242.

[3] The properties of linkages are much more vividly brought out, both for individual study and for demonstration to a class, by working models than by mere diagrams. Such models are very easily made, after a little practice, with the links of moderately stiff cardboard, pivoted together with eyelets by means of an eyelet punch and set.

and from this the construction, afterward called the parallel motion, was derived. . . . Though I am not over anxious after fame, yet I am more proud of the parallel motion than of any other invention I have ever made." The whole curve described by E is a figure eight sextic, with the two parts through the node deviating very slightly from a straight line.

Watt applied this to his engines, using the walking-beam for one of the fixed arms, and adding two more links in a pantograph arrangement so that a second point was made to move in a straight line parallel to that described by E. One of these points was attached to the piston-rod and the other to the air-pump rod.[1] It is probably this arrangement that was the origin of the name "parallel motion," which has been carried over, though obviously a misnomer, to the simpler form and to all other linkworks designed to give rectilinear motion to a point.

There does not seem to have been any scientific investigation of the subject during the first half of the nineteenth century. The first mathematician to take it up was Tchebychef (variously spelled as romanized in different countries), professor in the University of St. Petersburg. He became interested in it while giving a course in applied mechanics, and in 1852 took a trip[2] to western Europe, especially to visit factories, see different types of mechanism, and most of all to study the Watt parallel motion. In England he hunted up Watt's original machines,[3] wherever he could find them, in order to ascertain the lengths of the various parts. On his return, he studied the question of getting the closest possible approximation to rectilinear motion by linkwork. This led to an analytical problem which he treated skilfully and at great length, inventing new methods of analysis which are beyond the scope of this paper. His results enabled him to discuss the Watt movement, and to devise new ones, all of them only approximately straight line motions, but in every case accompanied by formulas showing the amount of deviation from a straight line as a function of the arbitrary constants of the linkwork. This deviation, by a proper choice of the constants, can be made very small, well within the limits of mechanical accuracy in the construction, so that for practical purposes they are quite as useful as if theoretically exact.

Tchebychef had considered the problem of obtaining exact rectilinear motion, and had finally come to the conclusion that it was impossible, although he did not succeed in proving this. In 1871, however, one of his own students, Lipkin, discovered a seven-bar linkwork which would describe an exactly straight line. The Russian government awarded him a prize, and then it transpired that identically the same device had been discovered by a French officer, Peaucellier, in 1864, and announced, but not described, in the *Nouvelles Annales*[1] of that year, without attracting much attention. Peaucellier's priority and Lipkin's independence were both fully established.

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[1] See the *Century Dictionary*, 1911, p. 4278, under "parallel" for picture of application to walking-beam.

[2] For his own interesting account of the trip, see *Œuvres de P. L. Tchebychef* [in French], St. Petersburg, 1899, Vol. II, pp. vii–xviii. The various papers giving his work on linkages and the functions associated with them will be found in the two volumes. A brief general survey of his life and work, and a special account of his results on linkages, are contained in *P. L. Tschebyschef und seine wissenschaftlichen Leistungen*, Wassilief and Delaunay, Teubner, 1900.

[3] A very interesting collection of original engines and models made by James Watt and his workmen may now be seen in the Science Division of the South Kensington Museum, London.

This seven-bar linkwork (Fig. 2) consists essentially of a six-bar linkage, called the Peaucellier cell, and an extra link. The cell is composed of a rhombus, P_1MP_2M', and two equal bars, OM, OM', pivoted to opposite vertices and pivoted together at O. By symmetry the three points O, P_1, P_2 (respectively called the fulcrum, first pole, and second pole) are always collinear, and by simple geometry:

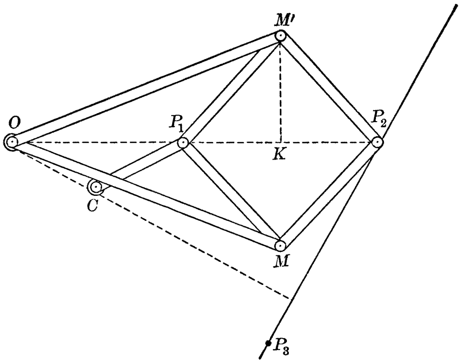

FIG. 2.

$$\overline{OP}_1\cdot\overline{OP}_2 = (\overline{OK} - \overline{P_1K})\cdot(\overline{OK} + \overline{P_1K}) = \overline{OK}^2 - \overline{P_1K}^2$$

$$= (\overline{OM}^2 - \overline{KM}^2) - (\overline{P_1M}^2 - \overline{KM}^2) = \overline{OM}^2 - \overline{P_1M}^2, \text{ a constant.}$$

That is, the product of the distances of the two poles from the fulcrum is constant. Thus, keeping O fixed, if P_1 describes any curve, P_2 will describe the inverse curve. It is a well-known fact, which may be shown by elementary geometry, that the inverse of a circle is in general a circle, but when the center of inversion lies on the original circle, the inverse is a straight line. Hence by constraining P_1 to move in a circle through O, which may be done by pivoting O to a fixed plane, and adding another link, CP_1 pivoted at C so that $CP_1 = OC$, we may make P_2 move in a straight line P_2P_3, perpendicular to OC. Or, by making $CP_1 \neq OC$, we may make P_2 move in a circular arc of any desired radius, by properly adjusting the constants.

This beautiful device aroused much interest in the study of linkages in general, and during the five years beginning 1874 more than one hundred[2] papers on the subject appeared in the mathematical journals. Various mathematical applications and generalizations of the Peaucellier apparatus were made. In mechanism, comparatively little use seems to have been made of it, for other methods of getting rectilinear motion are more convenient in practice. Many special linkworks have been constructed to describe different curves, such as conics, cardioids, limaçons, lemniscates, cissoids, etc. The simplest conicograph contains seven bars. Linkages have been applied to complex variables and elliptic functions, and to realizing numerous algebraic transformations.[1]

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[1] 2^e^ série, III, p. 414.

[2] Liguine, *Liste de travaux sur les systèmes articulées*, *Bull. des Sci. Math.*, 2e série, VII, pp. 145–160, gives 150 titles up to 1882. For later bibliography see *Royal Society Catalogue of Scientific Papers, 1800–1900, Subject Index,* Vol. II, *Mechanics*, pp. 84–86.

The study of the curves described by a linkwork can most readily be made by the use of polar coördinates, which may later be transformed into Cartesian.[2] In general, a three-bar linkwork describes a sextic curve, and in general the curves described by any linkwork turn out to be algebraic. This and its converse were stated by Peaucellier[3] from general reasoning, which did not satisfy Sylvester, who outlined a proof. It remained for Kempe to prove explicitly: "A linkwork can be found to describe any given algebraic curve." There is no space here to outline his simple and elegant proof.[4] The method is much more complicated than most of the special linkworks devised for important curves, but it is of theoretical value. The opposite theorem is also true, that no transcendental curve can be described by a linkwork. These results definitely show us the large possibilities of linkworks, and also their limitations.

THE PROBABLE RANK IN A LARGE CLASS OF A STUDENT OF GIVEN RANK IN A SMALL CLASS.

By L. D. AMES, University of Missouri.

In a certain statistical investigation it was desirable to compare students in different classes on the basis of their ranks in their respective classes. To reduce the two rankings to a common unit we seek to know the probable rank in a very large class of a student who ranks kth in a class of n students.

For example, we may wish to know how a given class of students who have had a certain definite type of training compare in their subsequent work with the average of students with whom they may compete. We follow these particular students into various other classes and find, for example, that one of them ranks second in a class of nine, another ranks sixth in a class of ten, etc. We wish to assign a numerical value to these rankings. Does the second student in a class of four probably rank, other things being equal, on a par with the twenty-second student in a class of forty-four?

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[1] See Emch, *An Introduction to Projective Geometry and Its Applications*, Wiley, 1905, for a chapter on linkage transformations and references to original papers.

[2] For examples of the methods of attack, see F. Dingeldey, *Über die Erzeugung von Kurven vierter Ordnung durch Bewegungsmechanismen*, Teubner, 1885, Chapter III. Also Königs, *Leçons de cinématique*, Paris, 1897, Chap. XI.

[3] *Note sur une question de géometrie de compas*, *Nouvelles Annales*, 2^e série, XII, p. 71, 1873. This, by the way, is Peaucellier's first published description of his invention.

[4] *On a General Method of Describing Curves of the nth Degree by a Linkwork*. *Proc. of London Math. Soc.*, 1876, VI, pp. 213–216. Also given in Königs, *op. cit.*, pp. 269–273.